diagonal optimal solution can be attained. Certainly, this design need pay the penalty of the higher-order controller.

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Optimal Transfer from Collinear Libration Points with Limited Rotation Speed

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I. Introduction

THE papers[2-5] have dealt with the problem of reaching the collinear and equilateral libration points of the Earth–Moon system of a space vehicle that is placed in Earth or Moon orbit.

It has been seen, using the impulse method, that small variations in the initial velocity result in important deviations from the libration points. This disadvantage can be eliminated by the continuous use of a low thrust that leads to a large class of problems concerning several requirements of optimizing certain quantities, such as the fuel consumption and the period of transfer for evolutions of short periods.

The present paper studies the transfer in minimum time of a mass particle initially at rest at the libration point, using a small thrust with constant magnitude and bounded angular speed to achieve a prescribed final velocity. The restrictions divide the optimal trajectory into three arcs. On this trajectory, the rotation velocity of the direction of the propulsion has the extremal value or corresponds to Lawden's tangent law. The matching of the arcs together with transversality conditions and final conditions determines the constants of integration and the evolution time. The equations of motion are integrated from the assigned initial conditions. The final values on one arc are taken as initial falues for the next arc. Commutation times t_1 and t_2 of the control, the integration constants of the adjoint system, and the total time are determined from the continuity requirement at the end of the first arc and from the transversality and initial conditions.

II. Equations of Motion

In the phase space $X = \{x_i\}$ $(i = 1, \ldots, 5)$, the equations of motion in a rotating system of axes with the origin at the collinear libration points of a vehicle acted upon by a propulsion force of small magnitude may be written as[2]

$$\dot{x}_1 = x_2 \tag{1a}$$

$$\dot{x}_2 = K_1 x_1 + 2\omega x_4 + \alpha \cos x_5 \tag{1b}$$

$$\dot{x}_3 = x_4 \tag{1c}$$

$$\dot{x}_4 = -2\omega x_2 + K_2 x_3 + \alpha \sin x_5 \tag{1d}$$

$$\dot{x}_5 = u \tag{1e}$$

where x_{2k-1}, $x_{2k}(k = 1, 2)$ are the coordinates of the vehicle and, respectively, the components of the velocity; x_5 is the angle of orientation of the propulsion force with respect to a fixed direction in the present intertial system; α is the modulus of the propulsion force. As the angular velocity u of the propulsion vector is bounded, we have

$$|u| \leq \Omega_{\max} \tag{2}$$

III. Optimum Problem

Let us write the controlled system (1) in the form

$$\phi_i \equiv \dot{x}_i - f_i(x_1, \ldots, x_5, u) = 0 \qquad (i = 1, \ldots, 5) \tag{3}$$

where $x_i \epsilon X$ and $u \epsilon U$. The functions

$$f_i(x_i, u) \qquad \frac{\partial o f_i(x_i, u)}{\partial x_i} \qquad (i,j = 1, \ldots, 5) \tag{4}$$

are defined and continuous as $X \times U$. Consider two manifolds $S_1(t)$ and $S_F(t)$ defined by the equations

$$S_1 = \{x_i(0) = 0, \qquad i = 1, \ldots, 5\}$$

$$S_F = \{x_2(T) = c; \qquad x_4(T) = d; \qquad x_5(T) = \varphi\} \tag{5}$$

Among the admissible controls $u \epsilon U = \{u(t), \ |u(t)| \leq \Omega, \ 0 \leq t \leq T\}$ that transfer the representative point from the position $x_2 \epsilon S_1(t_2)$ at position $x_F \epsilon S_F(t_F)$, one has to determine those that minimize the functional

$$J = \int_0^T \mathrm{d}t \tag{6}$$

IV. Matching Conditions and Determination of Extremals

The Hamiltonian may be written as

$$H(x, u, \psi) = -1 + \sum_{i=1}^{5} \psi_i x_i \tag{7}$$

and let

$$M(x, \psi) = \max_{u \epsilon U} H(x, p, u) \tag{8}$$

We note that $H = H_{\max}$ with respect to u if

$$u = \Omega_{\max} = \Omega \qquad \text{for} \qquad \psi_5 > 0 \tag{9}$$

The adjoint system

$$\psi_i = -\frac{\partial H}{\partial x_i} \qquad (i = 1, \ldots, 5) \tag{10}$$

is written as

$$\dot{\psi}_1 = \psi_2 K_1 \tag{11a}$$

$$\dot{\psi}_2 = \psi_1 + 2\omega\psi_4 \tag{11b}$$

$$\dot{\psi}_3 = -K_2 \psi_4 \tag{11c}$$

$$\dot{\psi}_4 = 2\omega\psi_2 - \psi_3 \tag{11d}$$

$$\dot{\psi}_5 = \alpha(-\psi_2 \sin x_5 + \psi_4 \cos x_5) \tag{11e}$$

The characteristic equation attached to the differential

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system (11) is a two-quadratic equation whose roots are

$$\lambda_{1,2} = \pm a \qquad \lambda_{3,4} = \pm i\ell \tag{12}$$

where

$$a = \left[\frac{K_1 + K_2 - 4\omega^2 + \sqrt{(K_1 + K_2 - 4\omega^2 - 4K_1K_2}}{2}\right]^{1/2} \tag{13a}$$

$$\ell = \left[-\frac{K_1 + K_2 - 4\omega^2 - \sqrt{(K_1 + K_2 - 4\omega^2)^2 - 4K_1K_2}}{2}\right]^{1/2} \tag{13b}$$

Let $\bar{\psi}_i$ be the particular solutions of system (11). By means of the fundamental system of solutions

$$\psi_1 = \bar{\psi}_1; \qquad \psi_2 = \bar{\psi}_2; \qquad \psi_3 = \frac{\bar{\psi}_3 + \bar{\psi}_4}{2}; \qquad \psi_4 = \frac{\bar{\psi}_3 - \bar{\psi}_4}{2i} \tag{14}$$

We obtain the general solution of the adjoint system

$$\psi_1(t) = \psi_1^0 a(4\omega^2 - K_2 + a^2)e^{at} - \psi_2^0 a(4\omega^2 - K_2 + a^2)e^{-at} - \psi_3^0 \ell(4\omega^2 - K_2 - \ell^2)\sin\ell t + \psi_4^0 \ell(4w_2 - K_2 - \ell^2)\cos\ell t \tag{15a}$$

$$\psi_2(t) = \psi_1^0(-a^2 + K_2)e^{at} + \psi_2^0(-a^2 + K_2)e^{-at} + \psi_3^0(\ell^2 + K_2)\cos\ell t + \psi_4^0(\ell^2 + K_2)\sin\ell t \tag{15b}$$

$$\psi_3(t) = \psi_1^0(2\omega K_2)e^{at} - \psi_2^0(2\omega K_2)e^{-at} - \psi_3^0(2\omega K_2)\cos\ell t - \psi_4^0(2\omega K_2)\sin\ell t \tag{15c}$$

$$\psi_4(t) = \psi_4^0(2\omega a)e^{at} - \psi_2^0(2\omega a)e^{-at} - \psi_3^0(2\omega\ell)\sin\ell t + \psi_5^0(2\omega\ell)\cos\ell t \tag{15d}$$

Integrating the last relation from (11) and using the condition $\psi_5(0) = \psi_5^0$, we obtain on the first segment of the trajectory

$$\psi_5(t) = \psi_1^0\beta_4 e^{at}\sin\Omega_1 t + \psi_1^0\beta_2 e^{at}\cos\Omega_1 t - \psi_2^0\beta_4 e^{-at}\sin\Omega_1 t + \psi_2^0\beta_2 e^{-at}\cos\Omega_4 t + \psi_3^0\beta_3\cos(\ell - \Omega_4)t + \psi_3^0\beta_4\cos(\ell + \Omega_4)t + \psi_4^0\beta_3\sin(\ell - \Omega_1)t + \psi_4^0\beta_4\sin(\ell + \Omega_4)t + \psi_5^0 - \beta_2\psi_1^0 - \beta_2\psi_2^0 - \beta_3\psi_3^0 - \beta_4\psi_3^0 \tag{16}$$

where we noted

$$\frac{a}{a^2 + \Omega_1^2}\left[a(-a^2 + K_2) - 2a\omega\Omega_4\right] = \beta_1;$$

$$-\frac{a}{a^2 + \Omega_1^2}\left[2\omega a^2 + \Omega_1(-a^2 + K_2)\right] = \beta_2$$

$$\frac{a}{2(\ell - \Omega_1)}\left(\ell^2 + K_2 - 2\omega\ell\right) = \beta_3;$$

$$-\frac{a}{2(\ell + \Omega_1)}\left(\ell^2 + K_2 + 2\omega\ell\right) = \beta_4 \tag{17}$$

In the case where the switching function $\psi_5 = 0$ since $\psi_5 = 0$ from the last relation (11), it follows from the tangent law that

$$\mathrm{tg}x_5 = \frac{\psi_4(t)}{\psi_2(t)} \tag{18}$$

From the transversality conditions, we have

$$\psi_1(T) = \psi_3(T) = 0 \tag{19}$$

Let us remark that we do not know $\psi_2(T)$ and $\psi_4 T$. Since $x_5(0)$ and $x_5(T)$ are known, from (18) it follows that the tangent law cannot be extended to the whole trajectory, namely, it does not hold at the ends of the trajectory. We have the following possibilities:

Variants	$0 \le t \le t_1$	$t_1 < t \le t_2$	$t < t \le T$
V_1	$u = +\Omega$	$\mathrm{tg}x_5 = \frac{\psi_4(t)}{\psi_2(t)}$	$u = +\Omega$
V_2	$u = +\Omega$		$u = -\Omega$
V_3	$u = -\Omega$		$u = +\Omega$
V_4	$u = -\Omega$		$u = -\Omega$

(20)

For the sake of simplicity, we write Ω_1 and Ω_2 for the angular velocity on the intervals 1 and 3, respectively. Since the system is autonomous, we have

$$M(x, \psi)\Big|_{t=0} = 0 \tag{21}$$

$\psi_5(t)$ is obtained by integrating Eq. (11) where we place $x_s(t) = \Omega_1 t$. The use of Weierstrass-Erdmann conditions at $t = t_1$ gives

$$\psi_5(t_1) = 0 \tag{22}$$

The solution of the system (1) is written in the form

$$x_1(t) = \sum_{j=1}^{4} R_{ij}(t)x_j(0) + \sum_{j=0}^{4}\int_0^t K_{ij}(t,\tau)u_j(\tau)\,\mathrm{d}\tau \tag{23}$$

where

$$u_j(\tau) = [0,\ \alpha\cos x_5(\tau); \quad 0; \quad \alpha\sin x_5(\tau)] \tag{24}$$

and $R_{ij}(t)$ is the matrix of the normal fundamental solution at $t = 0$ of the system (1) at $u_j = 0$. It follows that

$$R_{ij}(0) = \delta_{ij} \tag{25}$$

We also denoted

$$K_{ij}(t,\tau) = \sum_{m=1}^{4} R_{im}(t)R_{mj}^{-1}\tau \tag{26}$$

R_{mj}^{-1} being the elements of the inverse fundamental matrix. From Eq. (23), it follows that

$$x_i(t) = \sum_{j=1}^{4} R_{ij}(t)x_j(0) + \int_0^t [R_{i2}(t-\tau)u_1(\tau) + R_{i4}(t-\tau)u_2(\tau)]\mathrm{d}_\tau \tag{27}$$

Since $x_5(0)\epsilon\ S_1$ by integration, we have

$$x_5(t) = \Omega_1 t \tag{28}$$

and the solution of the system (1) on the first trajectory arc may be written as

$$x_i^1(t) = \alpha\int_0^t [R_{i2}(t-\tau)\cos\Omega_1\tau + R_i\tau(t-\tau)\sin\Omega,']\mathrm{d}\tau \tag{29}$$

On the second arc, we have the tanget law

$$x_5^2(t) = \mathrm{arc\ tg}\frac{\psi_4(t)}{\psi_2(t)}\beta(t) \tag{30}$$

such that the parametric equation for the second arc becomes

$$x_i^2(t) = \sum_{j=1}^{4} R_{ij}(t)x_j^1(t_1) + \alpha\int_0^t [R_{i2}(t-\tau)\cos\beta(\tau) + R_{i4}(t-\tau)\sin\beta(\tau)]\mathrm{d}\tau \tag{31}$$

On the third arc, we have

$$x_5 = \Omega_2 \tag{32}$$

hence,

$$x_5^3(t) = \Omega_2 t + C_5^3 \tag{33}$$

If we impose the conditions at the end of the second arc, the constant of integration

$$C_5^3 = \beta(t_2) - \Omega_2 t_2 \tag{34}$$

is determined. It follows that

$$x_5^3(t) = \Omega_2 t + \beta(t_2) - \Omega_2 t_2 = \gamma(t) \tag{35}$$

Similarly, with (31) we obtain

$$x_6^3(t) = \sum_{j=1}^{4} R_{ij}(t)x_j^2(t_2) + \alpha\int_{t_2}^t [R_{i2}(t-\tau)\cos\gamma(\tau) + R_{ij}(t-\tau)\sin\gamma(\tau)]\,\mathrm{d}\tau \tag{36}$$

Imposing the final conditions yields

$$\sum_{j=4}^{4} R_{2j}(T)x_j^2(t_2) + \alpha\int_{t_2}^T [R_{22}(t-\tau)\cos\gamma(\tau) + R_{24}(t-\tau)\sin\gamma(\tau)]\,\mathrm{d}\tau = c \tag{37a}$$

$$\sum_{j=1}^{4} R_{4j}(T)x_j^5(t_2) + \alpha\int_{t_2}^T [R_{42}(t-\tau)\cos\gamma(\tau) + R_{44}(t-\tau)\sin\gamma(\tau)]\,\mathrm{d}\tau = d \tag{37b}$$

$$\Omega_2 T - \beta(t_2) - \Omega_2 t_2 = \varphi \tag{37c}$$

The transition from the second interval to the third one does not require additional conditions because the integration constants are determined from the relations

$$\psi_5(t_2) = 0 \tag{38a}$$

$$\mathrm{tg}x_5(t_2) = \frac{\psi_4(t_2)}{\psi_2(t_2)} \tag{38b}$$

Making explicit (19), (21), and (22), we will obtain

$$\psi_1^0\left[\beta_1 e^{at_1}\sin\Omega_1 t_1 + \beta_2 e^{at_1}\cos\Omega_1 t_1 - \beta_2 - \frac{\alpha}{\Omega_1}(-a^2+K_2)\right] + \psi_2^0\left[-\beta_1 e^{-at_1}\sin\Omega_1 t_1 + \beta_2 e^{-at_1}\cos\Omega_1 t_1 - \beta_2 - \frac{\alpha}{\Omega_1}(-a^2+K_2)\right] + \psi_3^0\left[\beta_3\cos(\ell-\Omega_1)t_1 + \beta_u\cos(\ell+\Omega_1)t_1 - \beta_3 - \beta_4 - \frac{\alpha}{\Omega_1}(\ell^2+\Omega_1^2)\right] + \psi_4^0\left[\beta_3\sin(\ell-\Omega_1)t_1 + \beta_4\sin(\ell+\Omega_1)t_1\right] = 0 \tag{39a}$$

$$\psi_1^0 a(4\omega^2 - K_2 + a^2)e^{aT} - \psi_2^0 a(4\omega^2 - K_2 + a^2)e^{-aT} - \psi_3^0\ell \times (4\omega^2 - K_2 - \ell^2)\sin\ell T + \psi_4^0\ell(4\omega^2 - K_2 - \ell^2)\cos\ell T = 0 \tag{39b}$$

$$\psi_1^0 e^{aT} + \psi_2^{0-aT} + \psi_3^0\cos\ell T + \psi_4^0\sin\ell T = 0 \tag{39c}$$

The relations (39) form an algebraic system with the unknowns $\psi_K^0/\psi_1^0(K = 2, 3, 4)$ whose function expressions of t_1 and T are used in the developments that give the state variables on the second and third interval of time.

In order to calculate t_1, t_2, and T, we have to take into account the fulfillment of the condition $0 \leq t_1 \leq t_2 \leq T$. The unknowns are the five integration constants that come from the adjoint system and t_1, t_2, and T. In order to determine the eight unknowns, relations (37) and (39) are used. The determination of the time of optimal transfer will be done by considering all variants (20), among with the one that for T gives the smallest positive value is picked up. In this way, we obtain the succession of arcs of the optimal trajectory.

The motions with $u = \pm\Omega$, as well as those with $\mathrm{tg}x_5 = \psi_2(t)/\psi_4(t)$, are optimal. The foregoing analysis shows that the optimal trajectory consists of three arcs, among which those situated at the extremities are determined by the control function $u = \pm\Omega$. The intermediate portion corresponds to the tangent law. We have to note that a complete treatment that would permit one to calculate the time of the arcs of the optimal trajectory is possible only by numerical integration. For the simplified equations of motion of a material point,[1] the tangent law gives a constant value for the direction of the propulsion. Our analysis shows that the motion around the collinear libration pointe gives a variation in terms of time of the orientation angle of the propulsion that considerably complicates the problem. Thus, the determination of the unknowns implies the solution of some integral equations and of nonlinear algebraic systems.

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Minimum Impulse Orbital Evasive Maneuvers

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Introduction

MUCH research has been done in the dynamics of orbital transfer and rendezvous, but few papers have dealt with the problem of avoiding an interception in space. Many military capabilities are being put in satellites and the means of intercepting them are improving. Some research has been done

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